

# Kirchhoff's Voltage Law (KVL) and Faraday's Law: ElectroBOOM's Experiments

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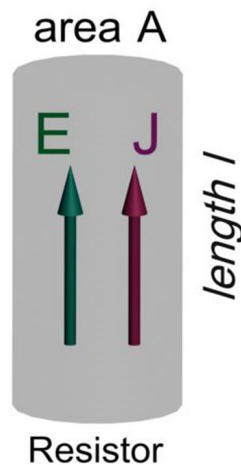
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## 1. Introduction

This document is an attempt to explain my understanding of the series of very nice experiments by Mehdi Sadaghdar, ElectroBOOM, see <https://www.youtube.com/watch?v=0TTEFF0D8SA>. I am grateful to Mr. Sadaghdar for a number of discussions about Faraday's Law and KVL, which have improved my understanding of both. Any mistakes in the discussion below are entirely mine, however.

I will go through the circuits he presents one at a time, and explain how what he is measuring fits into my conception of Faraday's Law, with some supplemental material along the way to help understand what I am saying.

## 2. Review of Ohm's Law in Macroscopic and Microscopic Form



First we need to understand what Ohm's Law tells us. There is a microscopic form and a macroscopic form, as explained below.

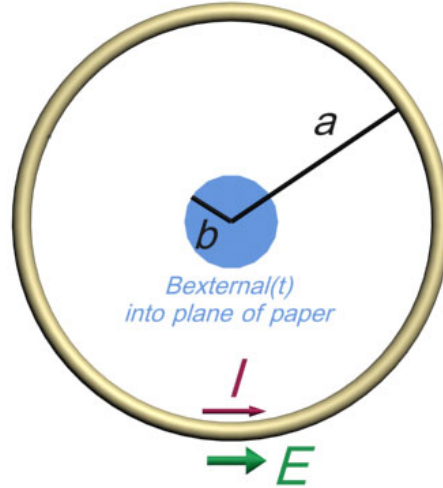
$$\mathbf{J} = \sigma_{\text{conductivity}} \mathbf{E} = \mathbf{E} / \rho_{\text{resistivity}} \quad \text{microscopic form}$$

$$|\mathbf{J}| = I / A \quad |\mathbf{E}| = V / l$$

$$\mathbf{J} = \mathbf{E} / \rho_{\text{resistivity}} \Rightarrow I / A = (V / l) / \rho_{\text{resistivity}}$$

$$\text{Or } V = I \left( \frac{l \rho_{\text{resistivity}}}{A} \right) = IR \quad \text{macroscopic form} \quad R = \left( \frac{l \rho_{\text{resistivity}}}{A} \right)$$

### 3. Uniform Toroidal Resistor in an External Magnetic Field, Ignoring Self-field



We have a torus made out of a wire with radius  $d$  bent into a circle of radius  $a$  (see sketch,  $a \gg d$ ), made of uniformly resistive material with resistivity  $\rho_{resistivity}$ . The resistance of this uniform toroidal resistor is

$$R = \left( \frac{l\rho_{resistivity}}{A} \right) \approx \left( \frac{2\pi a\rho_{resistivity}}{\pi d^2} \right).$$

It has an external magnetic field threading it which is given by

$$\mathbf{B}_{external}(t) = \begin{cases} -\hat{\mathbf{z}} B_o \frac{t}{T} & r < b \\ 0 & r > b \end{cases}$$

That is, the external field is increasing in time, pointed into the page, and is zero outside of a circle of radius  $b$ . This time changing magnetic field will produce an electric field in the wire which is everywhere the same. We can find this electric field using Faraday's Law, assuming we can ignore the magnetic field produced by the current flowing in the wire (we ignore the "self-field", and thus the self-inductance). This is ok because we can always make the resistance of the circuit high enough that the current through the circuit is small enough that the self-magnetic field is negligible compared to the external magnetic field. This gives

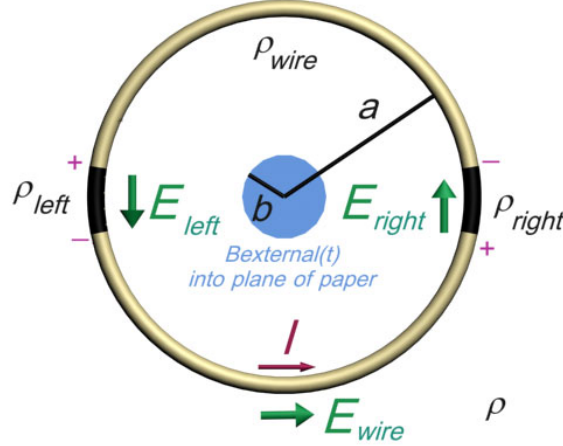
$$\oint_{\text{counterclockwise}} \mathbf{E} \cdot d\mathbf{l} = 2\pi a E = -\frac{d}{dt} \int \mathbf{B}_{external} \cdot \hat{\mathbf{z}} dA = \frac{d}{dt} \left( B_o \frac{t}{T} \pi b^2 \right) = + \frac{B_o \pi b^2}{T}$$

The plus sign in the equation above means that the electric field is everywhere counterclockwise around the torus (as shown in the sketch above) and is it given in magnitude by

$$E = \frac{B_o b^2}{2Ta}$$

The current in the uniform torus is given by Ohm's Law:  $I = \frac{V}{R} = \frac{2\pi a E}{R} = + \frac{B_o \pi b^2}{RT}$ .

#### 4. Non-Uniform Toroidal Resistor in an External Magnetic Field, Ignoring Self-field



Now we replace our uniform torus above with a non-uniform torus consisting of three different resistivities,  $\rho_{wire}$ ,  $\rho_{right}$ , and  $\rho_{left}$  as shown in the sketch, where we assume  $\rho_{wire}$  is much smaller than the other two resistivity's. Suppose the "height" of the sections with  $\rho_{right}$  and  $\rho_{left}$  is the same, say  $l$ . The total resistance of the torus is

$$R \approx \left( \frac{(2\pi a - 2l)\rho_{wire} + l\rho_{left} + l\rho_{right}}{\pi d^2} \right)$$

The current will be everywhere uniform around the torus (for an explanation of this see 5 below) and given, as before, by  $I = + \frac{B_o \pi b^2}{RT}$ . But now we will have three different values of the electric field in the different parts of the torus, given by

$$(2\pi a - 2l)E_{wire} = IR_{wire} = I \frac{(2\pi a - 2l)\rho_{wire}}{\pi d^2} \Rightarrow E_{wire} = \rho_{wire} \frac{I}{\pi d^2} = \rho_{wire} J$$

$$lE_{left} = IR_{left} = I \frac{l\rho_{left}}{\pi d^2} \Rightarrow E_{left} = \rho_{left} \frac{I}{\pi d^2} = \rho_{left} J$$

$$lE_{right} = IR_{right} = I \frac{l\rho_{right}}{\pi d^2} \Rightarrow E_{right} = \rho_{right} \frac{I}{\pi d^2} = \rho_{right} J$$

where our expression for the field in each region reduces to the microscopic form of Ohm's law on the far right of each equations, as we expect. The thing about this that surprises most people is that the electric field now varies in the torus, where in the uniform case it was uniform in the torus. How can this happen?

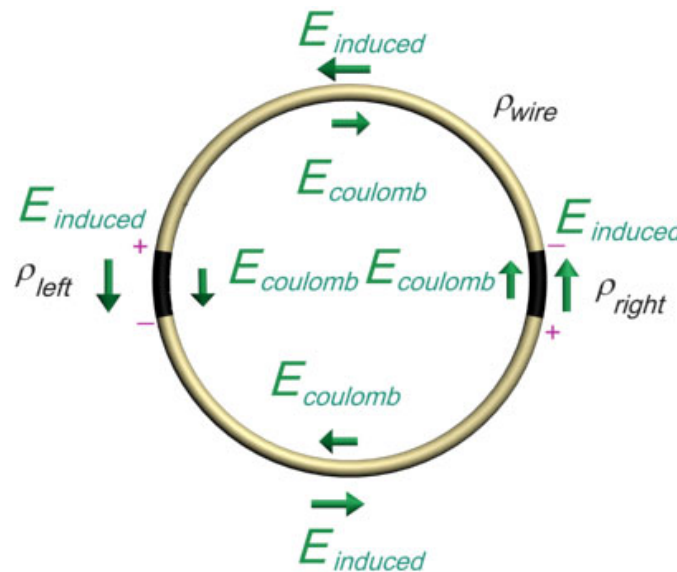
### 5. Why is the Current Uniform around the Non-Uniform Torus?

In the uniform example in 3 above we had no electric charge anywhere, and our electric field was just produced by a time changing magnetic field. But once we introduce variations in our resistivities, as we have done in 4 above, charges start building up in the circuit at the junctions between the different resistivity's. Why does this happen? Well suppose that at  $t = 0$  we turn on the magnetic field in the non-uniform torus and start increasing it with time. Just after we turn on the current, there will be everywhere in the circuit a uniform electric field given by our result in 3 above,  $E = \frac{B_o b^2}{2Ta}$ .

But that electric field in the wire will drive a much larger current in the wire than that same electric field in either one of the left or right resistors, because the current density will initially be

$$\mathbf{J} = \mathbf{E} / \rho_{wire} = \frac{B_o b^2}{2Ta} \frac{1}{\rho_{wire}} \gg \frac{B_o b^2}{2Ta} \frac{1}{\rho_{left}}, \frac{B_o b^2}{2Ta} \frac{1}{\rho_{right}}$$

since we are assuming  $\rho_{wire}$  is much smaller than the other two resistivities. Thus the *initial* currents just after  $t = 0$  are non-uniform around the circuit, but this does not last long. The initial imbalance in currents will have the following subsequent effects (note that the largest initial current is in the part of the circuit with the smallest resistivity, that is the wire). This much larger initial current in the wire will cause the top of the left resistor to start charging up positive, since there is a much larger current flow coming in from the top of the wire than is able to leaving through the left resistor.



The net effect is that this junction will rapidly charge up positive. Similar charges of various signs will build up at the other three junctions, for exactly the same reasons, as we show in the sketch.

Once these charges start building up, we now have both the electric field due to the time changing magnetic field (let's call it  $\mathbf{E}_{induced}$ ) and the electric field due to the charges (let's call it  $\mathbf{E}_{coulomb}$ ), with  $\mathbf{E} = \mathbf{E}_{coulomb} + \mathbf{E}_{induced}$ . In the wire the sum of these two fields will cause the total electric field to *decrease* (see sketch of top and bottom part of circuit above). In this sketch at the top we know that the coulomb electric field caused by the charges must point from the positive charge on the top of the left resistor to the negative charge on the top of the right resistor, with a similar argument about the direction of the coulomb electric field on the bottom. In both cases, the coulomb electric field in the wire is opposite the direction of the induced electric field (but smaller in magnitude), and hence the electric field in the wire is reduced. By Ohm's Law, that will reduce the flow of current in the wire.

In contrast, within the resistors, in the left and right resistors the two fields (induced plus coulomb) are in the same direction and this will cause the total electric field in the left and right resistors to *increase* over its initial value. By Ohm's Law, this will increase the flow of current through the resistors in the wire. Thus what initially started as much larger current in the wire and much smaller current in the resistors leads to a build-up in charge at the junctions that tends toward reducing the current in the wire and increasing the current in the resistors. This process continues until the current flow throughout the circuit is everywhere the same.

That is, this charging will continue until we get to the point where the amount of current flowing into any junction is the same as the current flow out, and we have reached equilibrium, so that there is no more charge build up. This will leave us with no electric field in the wire, but electric fields in both resistors, and these are the electric fields we found above in 4, where we begin with the assumption that the current is the same everywhere in the circuit. This is a true statement, and this approach to the same current everywhere in the circuit happens very fast, too fast for us to realistically measure it.

In the limit that  $\rho_{wire} = 0$ , the electric field is zero in the wires (even though current still flows there, since it takes only a zero electric field to get a current to flow if there is zero resistance to the flow of current). In this limit, we have

$$E_{wire} = 0$$

$$E_{left} = \rho_{left} \frac{I}{\pi d^2} = \rho_{left} J \Rightarrow lE_{left} = I \frac{l\rho_{left}}{\pi d^2} = IR_{left}$$

$$E_{right} = \rho_{right} \frac{I}{\pi d^2} = \rho_{right} J \Rightarrow lE_{right} = I \frac{l\rho_{right}}{\pi d^2} = IR_{right}$$

$$lE_{left} + lE_{right} = IR_{left} + IR_{right} = I(R_{left} + R_{right}) = \frac{B_o \pi b^2}{T}$$

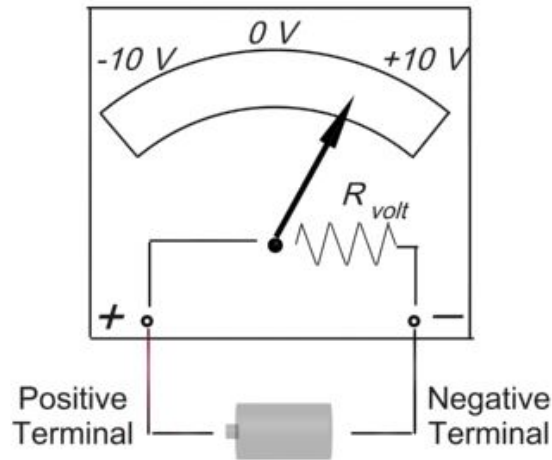
Our current is as before  $I = \frac{B_o \pi b^2}{RT} = \frac{B_o \pi b^2}{T(R_{left} + R_{right})}$ .

We can also derive an expression for the charges that must be at the junctions. For example on the top of the left resistor let there be a positive charge  $Q_+$ . The electric field in the left resistor is

$$E_{left} = \frac{IR_{left}}{l} = \frac{R_{left} B_o \pi b^2}{Tl(R_{left} + R_{right})} \text{ and } Q_+ = \pi d^2 \epsilon_o E_{left} = \pi d^2 \epsilon_o \frac{R_{left} B_o \pi b^2}{Tl(R_{left} + R_{right})}$$

with similar expressions for the other charges.

## 6. What Does a Voltmeter Measure?



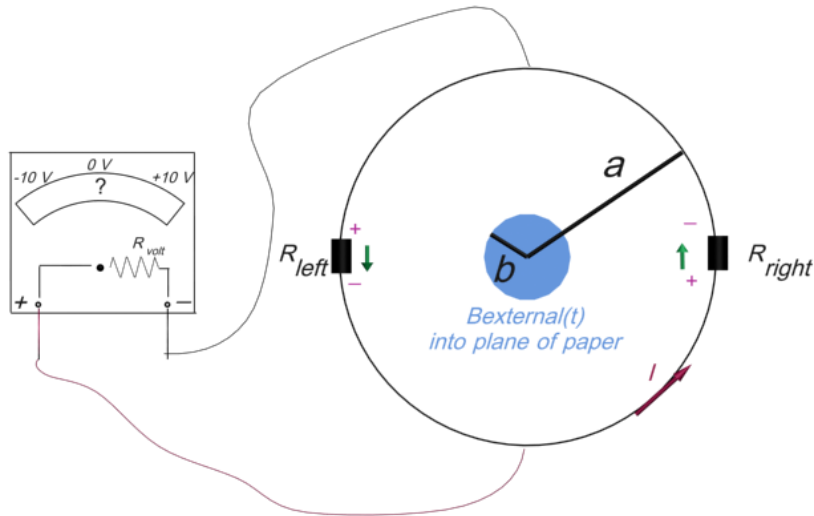
In the sketch above we are measuring the voltage of the battery connected as shown. The amount of the needle deflection measures the current through the voltmeter, and the scale is calibrated to read  $I_{\text{voltmeter}} R_{\text{voltmeter}}$ , where  $R_{\text{voltmeter}}$  is the internal resistance of the voltmeter. If current flows through the voltmeter from the “positive terminal” of the voltmeter to the “negative terminal”, then one reads a positive voltage (deflection of the needle clockwise). Generally  $R_{\text{voltmeter}}$  is chosen to be much greater than any resistance in the circuit, and voltmeters are put in parallel with circuit elements to measure the voltage across them. What one is measuring with the voltmeter is the voltage drop across the internal resistance of the voltmeter using Ohm’s Law,  $V = I_{\text{voltmeter}} R_{\text{voltmeter}}$ . In the sketch above we will measure a positive voltage if we hook up the terminals of a battery as shown. If we reverse the battery we will measure a negative voltage.

## 7. Measuring the Voltage in a Non-Uniform Toroidal Resistor in an External Magnetic Field, Ignoring Self-field

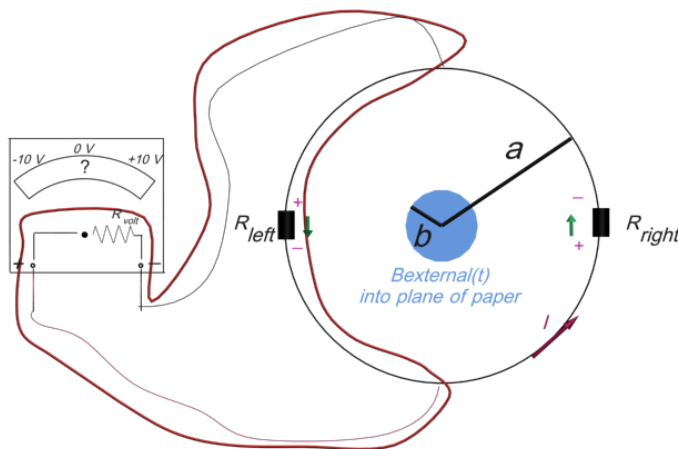
### a. First Configuration of Voltmeter

We make voltage measurements using a voltmeter connected to our circuit in the way shown in the sketch below. We only have three electric fields that are non-zero: in the resistors to the left and right and in the resistor in the voltmeter. What will the voltmeter measure? We will figure that out using

$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{\mathbf{n}} da$ . This law is true for any surface with its accompanying bounding contour, even for open surfaces whose contours have nothing to do with the wires or elements in the circuit. But of course to use this equation to solve problems, we need to conform the contour to the wires and elements in the circuit. We have two choices here of how to go around the circuit.



The first choice is as shown below. We go through the voltmeter and then through the left side of the torus, going clockwise around the contour (in the broad sense). Along that contour there are only two electric fields, the one in the voltmeter and the one in the left resistor. And there is no changing external magnetic flux through that open surface bounded by this contour. So  $\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{\mathbf{n}} da$  gives us  $I R_{\text{voltmeter}} + I R_{\text{left}} = 0$ . So the voltage we measure with the voltmeter is  $I R_{\text{voltmeter}} = -I R_{\text{left}}$ .





But we can also take a different path, and choose to go around the other side of the torus. When we do this we have (going around clockwise again in the general sense)

$$\oint \mathbf{E} \cdot d\mathbf{l} = IR_{\text{voltmeter}} - IR_{\text{right}} = -\frac{d}{dt} \int B \cdot \hat{\mathbf{n}} da = -\frac{B_o \pi b^2}{T}$$

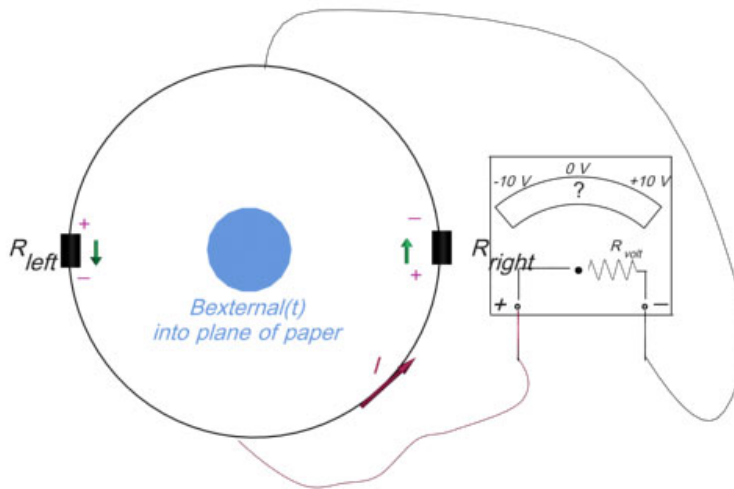
But if we remember that  $I = \frac{B_o \pi b^2}{(R_{\text{left}} + R_{\text{right}})T}$  then we see this is

$$IR_{\text{voltmeter}} - IR_{\text{right}} = -\frac{B_o \pi b^2}{T} = -I(R_{\text{left}} + R_{\text{right}}) \text{ or } IR_{\text{voltmeter}} = -IR_{\text{left}}$$

Thus we see that we get the same answer we got using the other path, which we must, otherwise something is seriously wrong with our physics (or mathematics).

### b. Second Configuration of Voltmeter

Now we move the voltmeter to a different configuration, as shown below. Note that we still have the positive terminal of the voltmeter connected to the bottom of the loop, as before, we have just shifted the position of the voltmeter and rearranged the leads.



Again, we have two choices here of how to go around the circuit. First let's go through the right side of the torus, so that we have a path that contains an open surface through which there is no changing external magnetic field. We get for this path, moving counterclockwise in the general sense

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int B \cdot \hat{\mathbf{n}} da = 0 = -IR_{\text{right}} + IR_{\text{voltmeter}} \text{ or } IR_{\text{voltmeter}} = IR_{\text{right}}$$

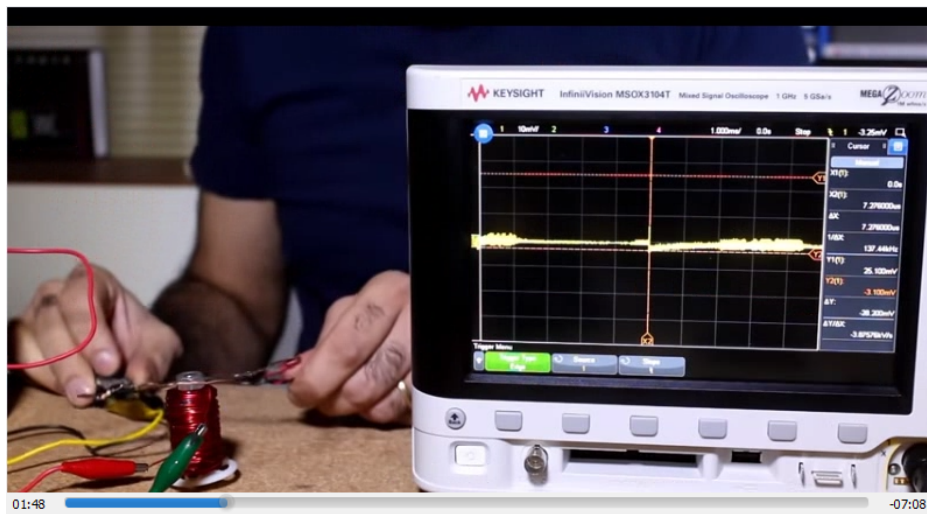
Thus in this case we measure a positive voltage on the voltmeter. We can also choose to go around the left side of the torus, in which case we get

$$\oint \mathbf{E} \cdot d\mathbf{l} = +IR_{left} + IR_{\text{voltmeter}} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{\mathbf{n}} da = +\frac{B_o \pi b^2}{T} = I(R_{left} + R_{right}) \text{ or } IR_{\text{voltmeter}} = IR_{right}$$

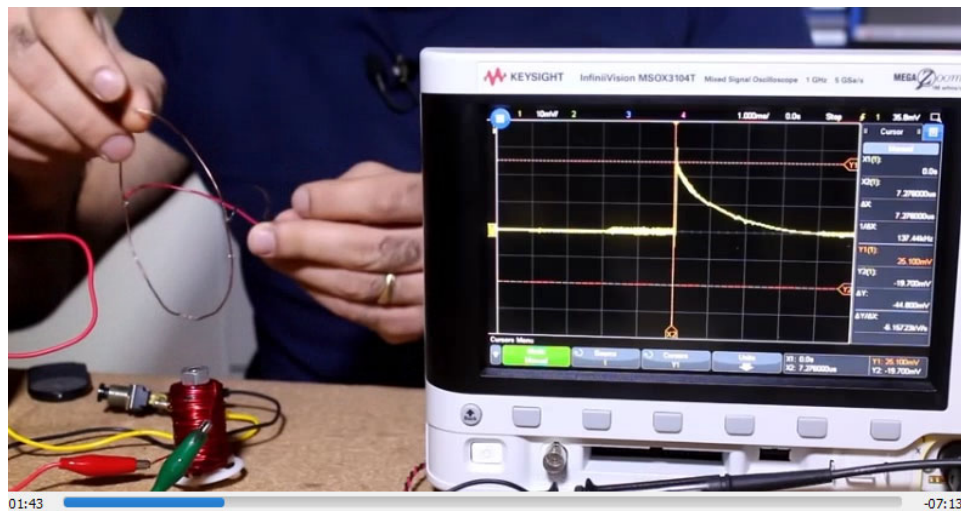
This is the same answer we got with the other loop.

### c. Correspondence to Mehdi's Video

Now let's discuss specific frames in Mehdi's video that correspond to the above calculations. To correspond to Mehdi's video, we need to take  $R_{left}$  to be his 1 kilo-ohm resistor and  $R_{right}$  to be his 10 kilo-ohm resistor. The frame just below is when he had the circuit configuration as in part 7a just above, where he measured a negative and relatively small voltage drop.



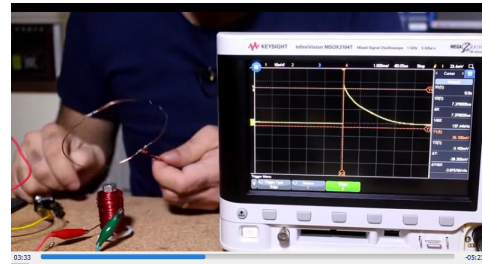
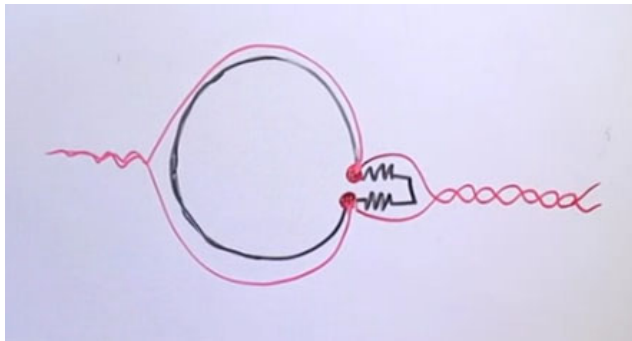
When Mehdi switched his circuit and measured the voltage in the new configuration, he measured a much larger positive voltage, which corresponds to the frame below. This is configuration 7b above.



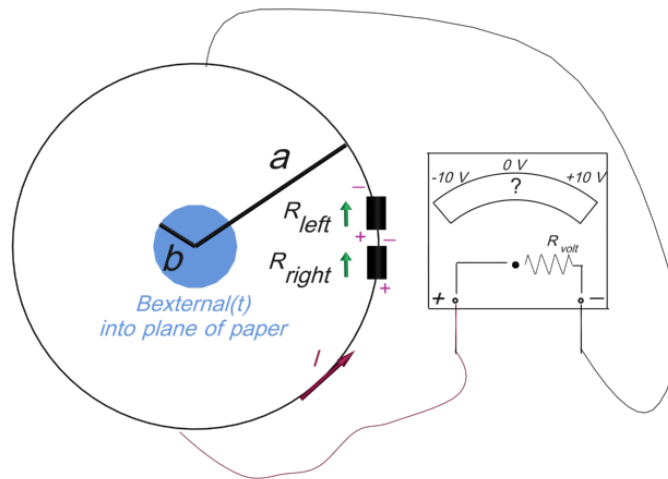
## 8. Another Circuit

### a. The Voltmeter on the Right

Now let's move on to another circuit that Mehdi considers. To the left below is his diagram for the circuit and the two ways he connects the voltmeter to this circuit. When the voltmeter is on the right Mehdi measures on the voltmeter  $IR_{\text{voltmeter}} = I(R_{\text{right}} + R_{\text{right}})$ , see the frame just below to the right.



Let's look at the circuit diagram for this setup. When we go around the circuit broadly clockwise going through the voltmeter but going on the right side of the torus, we get  $IR_{\text{voltmeter}} - I(R_{\text{left}} + R_{\text{right}}) = 0$

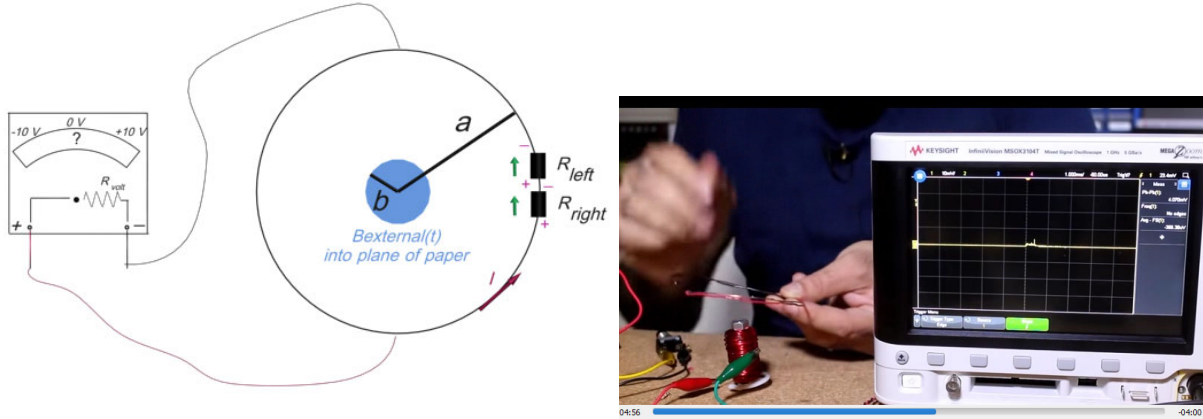


So we measure  $IR_{\text{voltmeter}} = I(R_{\text{left}} + R_{\text{right}})$ , in agreement with Mehdi's measurements. We can also choose to go through the voltmeter but going on the left side of the torus, which gives us

$$IR_{\text{voltmeter}} = \frac{B_o \pi b^2}{T} = I(R_{\text{left}} + R_{\text{right}}) \text{ as before.}$$

### b. The Voltmeter on the Left

When Mehdi instead puts the voltmeter to the left, he now measures zero voltage with the voltmeter (see frame to the right below). Let's look at the circuit diagram for this situation (diagram to the left below)

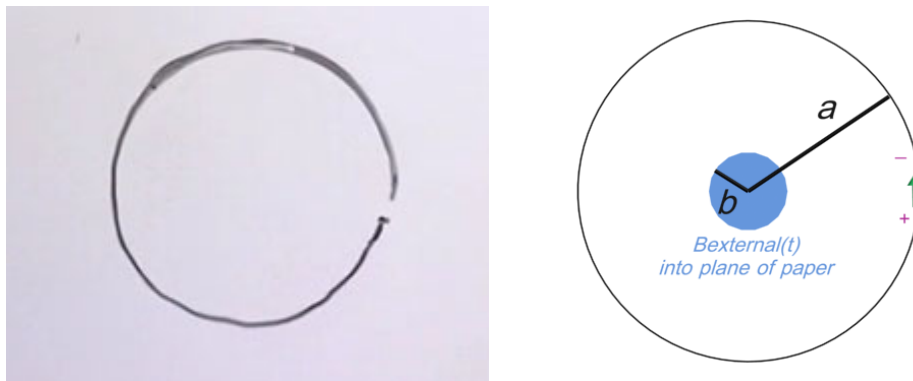


When we go around the circuit broadly clockwise going through the voltmeter going on the right side of the torus, we get

$$IR_{\text{voltmeter}} - I(R_{\text{left}} + R_{\text{right}}) = -\frac{B_o \pi b^2}{T} = -I(R_{\text{left}} + R_{\text{right}})$$

So we measure  $IR_{\text{voltmeter}} = 0$ , in agreement with Mehdi's measurements. We can also choose to go through the voltmeter but going on the left side of the torus, which gives us  $IR_{\text{voltmeter}} = 0$  as before.

## 9. Another Circuit



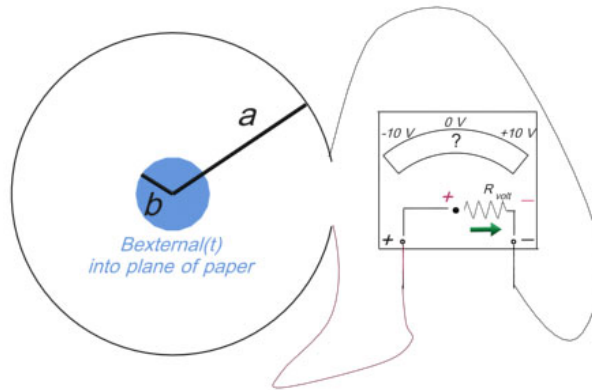
At 4:31 into the video, Mehdi draws the open circuit to the left above. There is no current flowing because the circuit is open. Let's discuss this situation before we put the voltmeter into the circuit. There will be zero electric field in the wires, because in the wires the induced electric field exactly cancels the coulomb electric field, as we saw before. But we will still see a charge accumulation on the ends of the wire, as shown in the schematic to the right above. The upper end of the wire will be

charged negatively and the lower end of the wire will be charged positively. Moreover we can calculate the potential difference across the gap by using Faraday's Law (which applies for any open surface and its bounding contour, where or not there are any wires around. Going around the circle counterclockwise gives us ( $l$  is the height of the gap)

$$\oint \mathbf{E} \cdot d\mathbf{l} = \int_{bottom}^{top} \mathbf{E}_{gap} \cdot d\mathbf{l} = lE_{gap} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{\mathbf{n}} da = +\frac{B_o \pi b^2}{T}$$

If you want to reassure yourself that this electric field actually exists, make  $\frac{B_o \pi b^2}{Tl}$  larger than the breakdown voltage in air, about a million volts per meter, and you will see a spark across the gap that proves that there is an electric field there, whether you put a voltmeter into the circuit or not.

### a. The Voltmeter on the Right



Now let's put a voltmeter in the circuit, as Mehdi does. First let's put it to the right, as shown above. We now apply Faraday's Law to the contour that goes around the torus and through the meter, since that is the only contour we have to choose from! The first thing to realize is that putting the voltmeter in the circuit allows charge to flow around the circuit and the positive and minus charges that were on the top and bottom ends of the wire have now disappeared, because we gave them a way to move. Where do they move to? They move to the ends of the voltmeter resistor, as shown above. At this point the only electric field along the contour bounding out open surface is in the resistor of the voltmeter, as shown. We can calculate the voltage drop across the resistor using Faraday's Law.

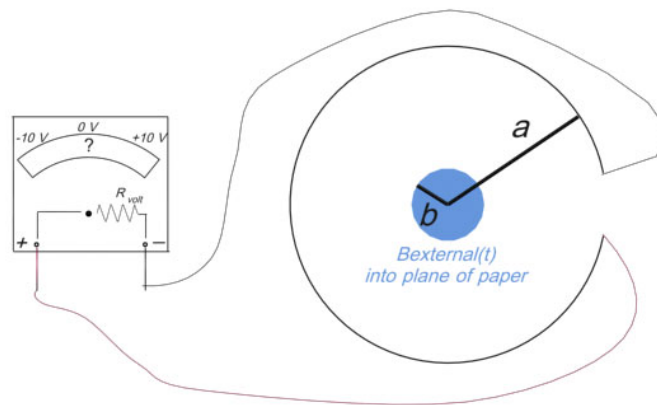
$$\oint \mathbf{E} \cdot d\mathbf{l} = IR_{voltmeter} = -\frac{d}{dt} \int \mathbf{B} \cdot \hat{\mathbf{n}} da = +\frac{B_o \pi b^2}{T}$$

So we measure a positive voltage, and even though the left and right resistors have disappeared from the circuit, we know that we had  $\frac{B_o \pi b^2}{T} = I(R_{left} + R_{right})$ , so we measure the same voltage difference we measured before in a different circumstance, even though the only resistor in the circuit is the

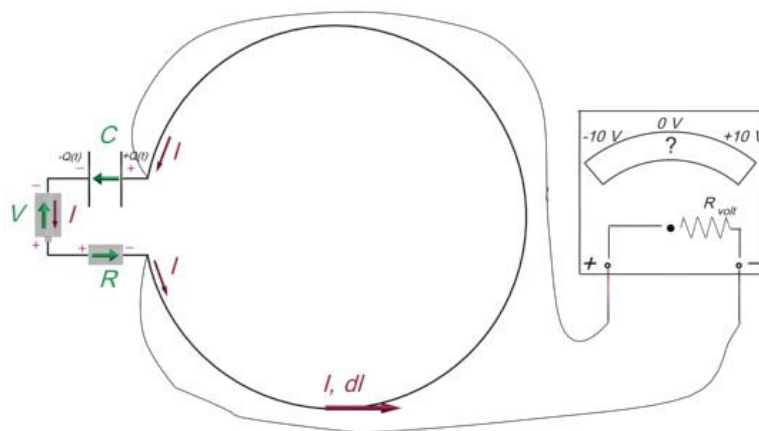
voltmeter resistor. Note that since  $R_{\text{voltmeter}}$  is really large, we don't have much current in the circuit, but it is not zero.

### b. The Voltmeter on the Left

Now let's put the voltmeter to the left. There again is only one contour, and that contour has no electric fields except (possibly) for one in the voltmeter, and also there is no changing magnetic flux through the open surface bounded by that contour. Thus we have  $\oint \mathbf{E} \cdot d\mathbf{l} = IR_{\text{voltmeter}} = 0$ . There is now no voltage across the voltmeter and no current in the circuit, period. And this is what Mehdi measures.



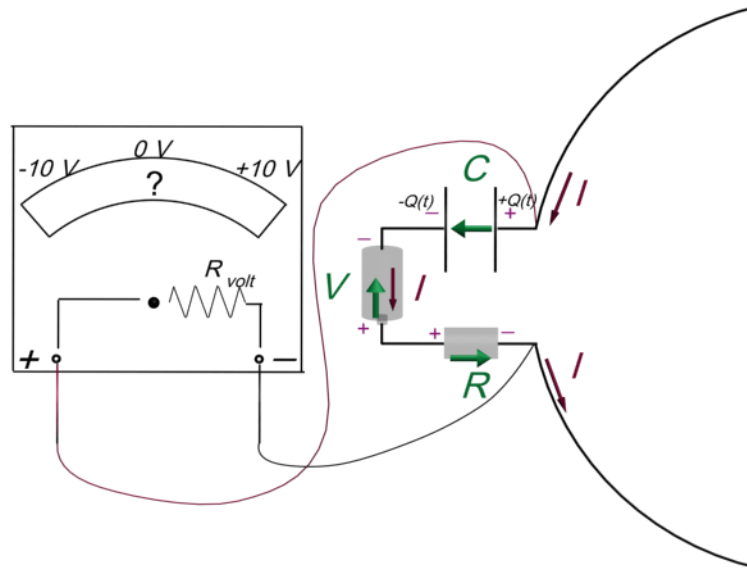
## 10. Another Circuit



The experiment above was not considered in the video, but I offer it up as an exercise for the reader. I have a "one-loop" inductor, with a capacitor and resistor, as shown, where I assume the self-magnetic field is only non-zero inside the loop. The battery is shown with the positive terminal down, which will result in a current flow counterclockwise around the circuit, giving a self-magnetic field out of the paper.

The inductance of the “one-loop” inductor is  $L$ . Show that if I hook up the voltmeter as shown, I read zero (or very small) voltage on the voltmeter. You can choose two ways to go around the circuit, either one should give you the same answer. (note: you can assume  $+V - IR - Q/C - L \frac{dI}{dt} = 0$ ).

In contrast, if you hook up the voltmeter as shown just below, the voltmeter will read  $-L \frac{dI}{dt}$ .



## 11 KVL

Many introductory texts on electromagnetism are not precise about what exactly they mean by the voltage drop across the inductor, and many students come to incorrect conclusions about what this actually means. The most common misconception is that the  $-L \frac{dI}{dt}$  voltage read by the voltmeter just

above represents a  $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$  through the inductor. But if the inductor wires are perfectly conducting, this integral is zero because there is no electric field in the wires.

One textbook that makes perfectly clear what the “voltage across the inductor” is is the textbook by Feynman, Leighton, and Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA 1964), Vol II, p22-2). In that textbook, the authors state explicitly that the  $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$  through the inductor must be zero, and they define the voltage difference across the inductor (which they correctly identify as an “electromotive force”, as  $\oint \mathbf{E} \cdot d\mathbf{l}$ , which by Faraday’s Law is  $-L \frac{dI}{dt}$ ). But this quantity, which has units

of volts, has nothing to do with  $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$  through the inductor, which is zero. Thus with Feynman et al.'s definition, the sum of all the voltage differences around the circuit is zero (that is, KVL holds)

$+V - IR - Q/C - L \frac{dI}{dt} = 0$ , but the first three terms here are the  $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$  through the various circuit

elements, and the last term has nothing to do with the  $-\int_a^b \mathbf{E} \cdot d\mathbf{l}$  through the inductor, which is zero. In

this sense, KVL holds, as argued by Mehdi Sadaghdar, but one must always remember that the voltage difference across the inductor is defined in a very different way compared to the voltage difference across the other three elements.

An excellent discussion of "electromotive force" (a terrible and misleading name) is given in the text by Griffiths, *Introduction to Electrodynamics 4th Edition*.